Block Codes with Embedded Quantization Step Size Information

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Abstract

If we quantize a block of n samples and then transmit information about quantization step size in the same bitstream, we may naturally expect such a code to be at least O(1/n) redundant. However, as we will show in this paper, this may not necessarily be true. Moreover, we prove that asymptotically, such codes can be as efficient as block codes without embedded step-size information. The proof relies on results from the Diophantine approximations theory. We discuss the significance of this finding for practical applications, such as the design of audio and video coding algorithms.

Background information and main result

Rate-distortion performance of scalar quantization

Let's assume that x is a random variable with Riemann-integrable density $p(x)$, and differential entropy

$$h(x) = -\int p(x) \log p(x)\, dx. \tag{1}$$

If we subsequently quantize x with a uniform quantizer with step-size Δ, we will arrive at a discrete random variable x^{Δ} with entropy

$$H\left(x^{\Delta}\right) = -\sum \Pr\left(x_i^{\Delta}\right) \log \Pr\left(x_i^{\Delta}\right) \tag{2}$$

As well-known from information theory (see, e.g., [1, Section 8.3]), the entropy of this new discrete variable x^{Δ} relates to the differential entropy of the original continuous variable x as follows:

$$H\left(x^{\Delta}\right) \to -\log(\Delta) + h(x). \tag{3}$$

This limit holds when $\Delta \to 0$.

Since the quantization step size Δ provides bounds for ℓ_∞-type distortion D_∞ :

$$\Delta/2 \ \leq D_\infty < \Delta, \tag{4}$$

the limit (3) also leads to the following simple approximation for the operational rate-distortion function of such a quantizer in a high-fidelity regime:

$$R(D_\infty) \sim -\log(D_\infty) + h(x). \tag{5}$$

The first term in this expression is the most important, as it captures the dependency on D_∞.

Block code with embedded quantization step size information

Let us now consider a case when we need to transmit a set of n samples $x_1, \dots, x_n$, produced by the source x, and we also need to transmit the information about the step size of Δ used to quantize them.

For instance, by assuming that $\Delta = \Delta(q)$, where q is a positive integer, we can construct code as follows:

1) Send prefix-code of integer q, and then
2) Send codes corresponding to samples $x_1, \dots, x_n$, quantized using step size $\Delta(q)$.

For a more specific example, we will further assume that

$$\Delta(q) = C/q, \tag{6}$$

where C is some positive constant.

We will next examine the rate-distortion performance of such block code.

Performance of a block code with the embedded quantization step size

We first note that the rate of transmission of q by any monotonic prefix code, such as Elias ω or Levenstein code [2], must be at least

$$R(q) \geq \log(q). \tag{7}$$

The rate of subsequent transmission of the quantized samples must follow (3,4).

By putting these two facts together, we can express the per-sample rate of this block code as follows

$$\begin{aligned} R_n &\geq \frac{1}{n}\log(q) - \log(\Delta) + h(x) \\ &= \frac{1}{n}\log\left(\frac{C}{\Delta}\right) - \log(\Delta) + h(x) \\ &= -\left(1+\frac{1}{n}\right)\log(\Delta) + h(x) + \frac{1}{n}\log(C). \end{aligned} \tag{8}$$

By further noting bounds for ℓ_∞ distortion (4), we arrive at the following estimate for the operational rate-distortion function in the high-fidelity regime:

$$R_n(D_\infty) \sim -\left(1+\frac{1}{n}\right)\log(D_\infty) + h(x) + \frac{1}{n}\log(C). \tag{9}$$

By comparing this estimate with (5), we conclude that the transmission of quantization step size information leads to a factor of $(1 + 1/n)$ increase in the principal ($O(\log(D_\infty))$) term of rate-distortion function. In other words, we observe a considerable loss in coding efficiency.

Of course, by making block size n larger, the relative overhead of the transmission of step size Δ can be made smaller, but this code construction also brings the following more fundamental questions:

- Since we transmit step size Δ, can we also choose it customarily for each given set of samples $\mathrm{x}_1, \dots, \mathrm{x}_\mathrm{n}$ to achieve a better approximation of all these values?
- Can such improved approximation errors be much smaller than the full step-size Δ?
- If the above is possible, can our block code perform much better than suggested by the above "naive" limit (9)?

To answer these questions, we bring the following fact from the Diophantine approximation theory [3].

Accuracy of simultaneous Diophantine approximations

Consider $n \geq 2$ irrational numbers $\xi_1, \dots, \xi_n$. Then [3, p.14, Theorem III]:

Fact 1. There exist infinitely many integers q and $p_1, \dots, p_n$, such that

$$\max_i \left| \xi_i - \frac{p_i}{q} \right| < \frac{1}{1+1/n} q^{-(1+1/n)}. \tag{10}$$

In other words, instead of a trivial limit $|\xi_i - p_i/q| < q^{-1}$, this result shows the existence of many solutions for which the approximation errors will be much lower.

To illustrate this phenomenon, in Figure 1, we plot approximations errors achieved with different values of q for two irrational numbers: $\xi_1 = \pi \approx 3.14159 \dots$ and $\xi_2 = e \approx 2.7182 \dots$. It shows a chain of "lucky" numbers q, for which approximation errors are much lower. Table 1 lists such best values q (found in the range $q \in [1,100]$), and the resulting rational approximations. Both trivial error limits (q^{-1}) and the actual maximum absolute errors are reported.

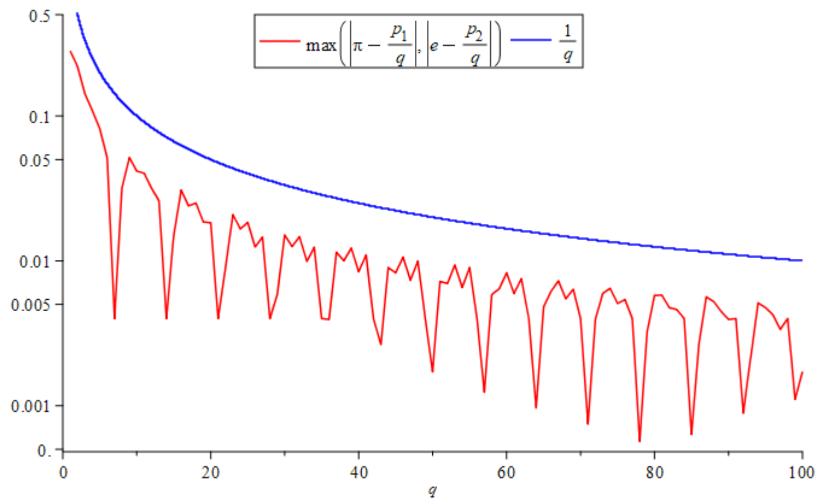

Figure 1. Accuracy of best rational approximation of two numbers $\{\pi, e\}$.

Table 1. Approximations of numbers: $\{\pi, e\}$ satisfying error bound (10).

q	p_1	p_2	q^{-1}	$\max\left\{\left\|\pi - \frac{p_1}{q}\right\|, \left\|e - \frac{p_2}{q}\right\|\right\}$
1	3	3	1	0.281718
2	6	5	0.5	0.218282
7	22	19	0.142857	0.003996
14	44	38	0.071429	0.003996
21	66	57	0.047619	0.003996
28	88	76	0.035714	0.003996
50	157	136	0.020000	0.001718
57	179	155	0.017544	0.001242
64	201	174	0.015625	0.000968
71	223	193	0.014085	0.000748
78	245	212	0.012821	0.000567
85	267	231	0.011765	0.000635

As it can be observed, in all cases listed in Table 1, the final approximation errors are much smaller than q^{-1} limits. We note that even some very small values q produce remarkably accurate approximations! For instance, in the above example, we note that $q = 7$ results in errors that are $O(10^{-3})$ small.

Of course, we must also note that the problem of finding such "lucky" numbers q is generally NP-complete [4]. However, as reported in [4, Section 5], with some relaxations on the bound (10), the problem can be reduced to a form allowing polynomial time solutions.

Achievable performance of code with embedded step size information

Let us now return to our problem of coding samples $x_1, \dots, x_n$. We note that with the use of a scalar quantizer with a step size $\Delta(q) = C/q$ and ℓ_∞-distortion metric, this problem becomes essentially equivalent to the problem of finding n simultaneous Diophantine approximation for numbers:

$$\xi_i = x_i/C, \qquad i = 1, \dots, n.$$

This observation suggests that for any given block of samples $x_1, \dots, x_n$, there should exist infinitely many quantization parameters q, for which the resulting ℓ_∞-type distortion satisfies

$$D_\infty < \frac{C}{1+1/n} q^{-(1+1/n)} = \frac{C^{-1/n}}{1+1/n} \Delta^{1+1/n}$$

By using this limit in our expression for block code length (8), and some simple algebra, we arrive at the following result:

Proposition 1. Given a block of samples $x_1, \dots, x_n$, there must exist infinitely many values of quantization sizes Δ, such that the resulting rate-distortion performance of a block code with embedded information about quantization parameter Δ satisfies:

$$R_n(D_\infty) \sim -\log(D_\infty) + h(x) + \mathrm{O}(1/n). \qquad (11)$$

This estimate holds in a high-fidelity ($\Delta \to 0$) regime.

We note that the principal ($O(\log(D_\infty))$) term in this expression has precisely the same factor as our original block code (5)! In other words, the overhead of the quantization step size transmission becomes fully compensated by the extra precision of approximations attainable by varying this parameter!

Applications and concluding remarks

The described coding problem is very common in many practical applications, such as audio and video coding [5,6]. In most cases, the quantization step size transmission is done to allow the encoder to achieve some target bitrate or some target quality while working with changing statistics of the source. A specific unit in such encoders, called "rate control" is typically responsible for adapting quantization step sizes dynamically, throughout duration of video or audio content, with the goal of achieving a certain given overall target. Furthermore, in implementations of such rate control logic, some assumptions are commonly made about the shape and continuity of the operational rate-distortion function $R(D)$ or rate-quantization-step-size function $R(\Delta)$ which is typically used to derive values of quantization parameter Δ to apply for the coded unit of data.

However, as we have shown in this paper, the related approximation problem is such that it is extremely naïve to expect continuity in distortion (and specifically in ℓ_∞-type distortion) with varying quantization step size Δ. As we have shown, with simultaneous rational approximations, the approximation errors can vary erratically with the increase in quantization step size, and in fact, there may be an infinite chain of step size values achieving much better rate-distortion tradeoffs.

In this context, the proposed mapping to the Diophantine approximations may become a useful tool for understanding of the nature of this problem, achievable performance limits, and for guiding towards suitable classes of solution funding algorithms.

References

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